FP–6 STREP 30717 PLATO–N (Aeronautics and Space)

PLATO–N

A PLAtform for Topology Optimisation incorporating Novel, Large-Scale, Free-Material Optimisation and Mixed Integer Programming Methods

Sequential Convex Programming Methods for Free Material Optimization

PLATO–N Public Report PU–R–4–2007

July 23, 2007

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Project co-funded by the European Commission within the Sixth Framework Programme (2002-2006)

Dissemination level: Public

Sequential Convex Programming Methods for Free Material Optimization

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July 23, 2007

Abstract:

We consider a numerical method for constrained nonlinear programming, that is widely used in mechanical engineering and that is known under the name SCP for sequential convex programming. The algorithm consists of solving a sequence of convex and separable subproblems, where an augmented Lagrangian merit function is used for guaranteeing convergence. Originally, SCP methods were developed in structural mechanical optimization, and are particularly applied to solve topology optimization problems. A new challenge for SCP methods is the solution of free material optimization (FMO) problems which contain additional semi-definite variables and even nonlinear semi-definite matrix constraints. A few formulations are investigated in more details and possible solution approaches are outlined.

Keywords:

Nonlinear Programming, sequential convex programming, method of moving asymptotes, free material optimization, semi-definite programming, interior point methods

1 Introduction

Sequential convex programming (SCP) algorithms are based on the observation that in some special cases, typical structural constraints become linear in the inverse variables. Although this special situation is rarely observed in practice, a suitable substitution by inverse variables depending on the sign of the corresponding partial derivatives and subsequent linearization is expected to linearize model functions somehow.

More general convex approximations are introduced by Svanberg [20] known under the name *method of moving asymptotes* (MMA). The goal is always to construct convex and separable subproblems, for which efficient solvers are available. Thus, we denote this class of methods by SCP, an abbreviation for *sequential convex programming*. The resulting algorithm is very efficient for solving mechanical engineering problems, if a proper starting point is available and if only a crude approximation of the optimal solution needs to be computed because of certain side conditions, for example calculation time or round-off errors in objective function and constraints. Some comparative numerical tests of SCP, SQP, and some other nonlinear programming codes are available for test problems from mechanical structural optimization, see Schittkowski, Zillober, and Zotemantel [17].

The computer code under investigation is the SCP routine SCPIP of Zillober [23, 21], an implementation of the method of moving asymptotes (MMA). Strictly convex and fully separable subproblems are solved by an interior point method combined with an active set strategy. Sparsity of the Jacobian matrix of the constraints is taken into account.

Topology optimization is one of the main areas, where SCP methods are applied. The idea is to distribute mass within a given volume, so that the global compliance of the structure is minimized. Since the number of the finite elements is often very big depending on the desired discretization accuracy, large nonlinear programs must be solved iteratively. The number of variables can be in the order of 10^5 to 10^6 or even more, see Zillober, Schittkowski and Moritzen [25]. In addition, more realistic structures require constraints for each element leading to a large number of nonlinear inequality constraints of the same order of magnitude.

In the subsequent section, we briefly outline SCP methods and especially the structure of the convex subproblems. Free material optimization is discussed in Section 3 where we show some proposals to handle semi-definite and diagonally dominant matrix variables depending on material properties. In Section 4 further nonlinear constraints for displacements, stresses, and global stability like buckling or vibration are introduced. The relationship to the PLATO-N software architecture and alternative approaches provided by the Technion is shown.

2 Sequential Convex Programming (SCP) Methods

We consider nonlinear optimization problems of the kind

$$
(\mathrm{P}) \left\{ \begin{array}{ll}
\min f(x) & \\
\text{s.t. } c_j(x) = 0 \, , & j = 1, \ldots, m_{\mathrm{eq}} \\
\qquad c_j(x) \leq 0 \, , & j = m_{\mathrm{eq}}+1, \ldots, m_c \\
\qquad x \in \mathbb{R}^n &
\end{array} \right.
$$

where $f(x)$ and $c_j(x)$, $j = 1, \ldots, m_c$, are twice continuously differentiable scalar functions.

Our goal is to create a sequence of convex and separable subproblems which are easy to solve due to their special structure, as illustrated in Figure 1. The corresponding solutions converge to the optimal solution of the original problem under some quite general assumptions.

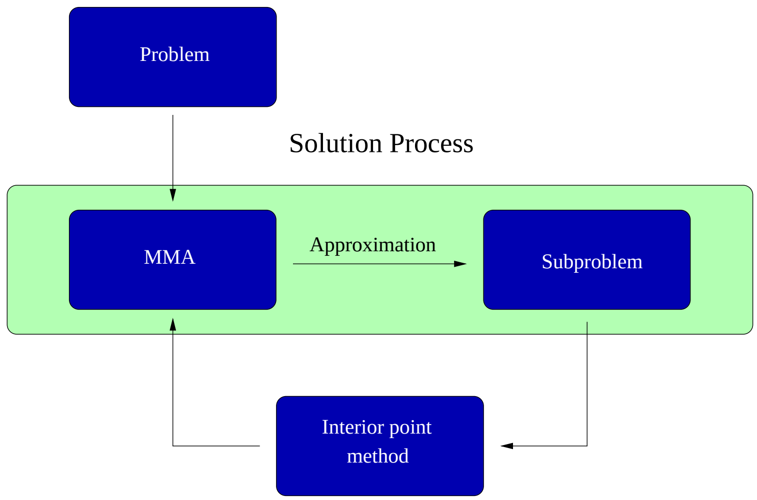

Figure 1: Procedure of MMA-Algorithm

One of the most important features of SCP methods based on the MMA technique is the introduction of two flexible asymptotes for each coefficient i of the optimization variable x^k in the k-th iteration, $i = 1, \ldots, n$. These asymptotes, an upper one U_i^k and a lower one L_i^k, reduce the feasible region and allow to control the curvature of a merit function, e.g., an augmented Lagrangian function, see Zillober [24]. Nonlinear equality constraints are linearized as in case of SQP methods. The strictly convex and separable subproblems possess diagonal Hessian matrices of the Lagrangian function which are explicitly known, and are solved by an interior point method taking sparsity patterns of the Jacobian matrix into account.

Objective function and the inequality constraints are linearized subject to the transformed variables $\dfrac{1}{U_i^k - x_i}$ or $\dfrac{1}{x_i - L_i^k}$, respectively. Thus, the resulting approximation of the objective function at the k-th iterate x^k is

$$\begin{aligned} f^k(x) \; := \; f(x^k) \; &+ \sum_{I_+^k} \left(\frac{\partial f(x^k)}{\partial x_i} \, (U_i^k - x_i^k)^2 \left(\frac{1}{U_i^k - x_i} - \frac{1}{U_i^k - x_i^k} \right) + \tau_i^k \frac{(x_i - x_i^k)^2}{U_i^k - x_i} \right) \\ &- \sum_{I_-^k} \left(\frac{\partial f(x^k)}{\partial x_i} \, (x_i^k - L_i^k)^2 \left(\frac{1}{x_i - L_i^k} - \frac{1}{x_i^k - L_i^k} \right) - \tau_i^k \frac{(x_i - x_i^k)^2}{x_i - L_i^k} \right) \end{aligned}$$

with $I_+^k := \{i| \dfrac{\partial f(x^k)}{\partial x_i} \geq 0\}$, $\quad I_-^k := \{i| \dfrac{\partial f(x^k)}{\partial x_i} < 0\}$, $\quad 0 < L_i^k < x_i < U_i^k$. τ_i^k are positive parameters, which are introduced to guarantee strict convexity of the approximated objective function. Equality constraints $c_j(x)$, $j = 1, \ldots, m_{\text{eq}}$ are linearized in the form

$$c_j^k(x) \; := \; c_j(x^k) \; + \; \sum_{i=1}^{n} \frac{\partial c_j(x^k)}{\partial x_i} \, (x_i - x_i^k)$$

for $j = 1, \ldots, m_{\text{eq}}$. The nonlinear inequality constraints $c_j(x)$ for $j = m_{\text{eq}} + 1, \ldots, m_c$ are approximated by

$$\begin{aligned} c_j^k(x) \quad := \quad c_j(x^k) \; &+ \sum_{I_+^{(j,k)}} \frac{\partial c_j(x^k)}{\partial x_i} \, (U_i^k - x_i^k)^2 \left(\frac{1}{U_i^k - x_i} - \frac{1}{U_i^k - x_i^k} \right) \\ &- \sum_{I_-^{(j,k)}} \frac{\partial c_j(x^k)}{\partial x_i} \, (x_i^k - L_i^k)^2 \left(\frac{1}{x_i - L_i^k} - \frac{1}{x_i^k - L_i^k} \right) \end{aligned}$$

with $I_+^{(j,k)} := \{i| \dfrac{\partial c_j(x^k)}{\partial x_i} \geq 0\}$, $I_-^{(j,k)} := \{i| \dfrac{\partial c_j(x^k)}{\partial x_i} < 0\}$ and the asymptotes are $0 < L_i^k < x_i^k < U_i^k$. With these definitions, we obtain the subproblem

$$
(P_{\text{sub}}^k) \begin{cases} \min f^k(x) \\ \text{s.t. } c_j^k(x) = 0\,, & j = 1, \ldots, m_{\text{eq}} \\ \quad c_j^k(x) \leq 0\,, & j = m_{\text{eq}}+1, \ldots, m_c \\ \quad \underline{x'} \leq x \leq \overline{x'}\,, \\ \quad x \in \mathbb{R}^n \end{cases}
$$

where $\underline{x'}$ and $\overline{x'}$ are suitable lower and upper bounds for x taking the asymptotes into account. This approximation of the nonlinear programming problem (P) is strictly convex and of first order, i.e.,

$$
\begin{aligned} f^k(x^k) &= f(x^k), & c_j^k(x^k) &= c_j(x^k), \\ \nabla f^k(x^k) &= \nabla f(x^k), & \nabla c_j^k(x^k) &= \nabla c_j(x^k), \end{aligned} \qquad j = 1,
$$

$\ldots$, m_c. The resulting subproblem (P_{sub}^k) is solved by an interior point method, where the structure of the diagonal Hessian matrices is exploited. The asymptotes are adapted in a special way, see Zillober [24] for more details and Figure 2 for an illustration.

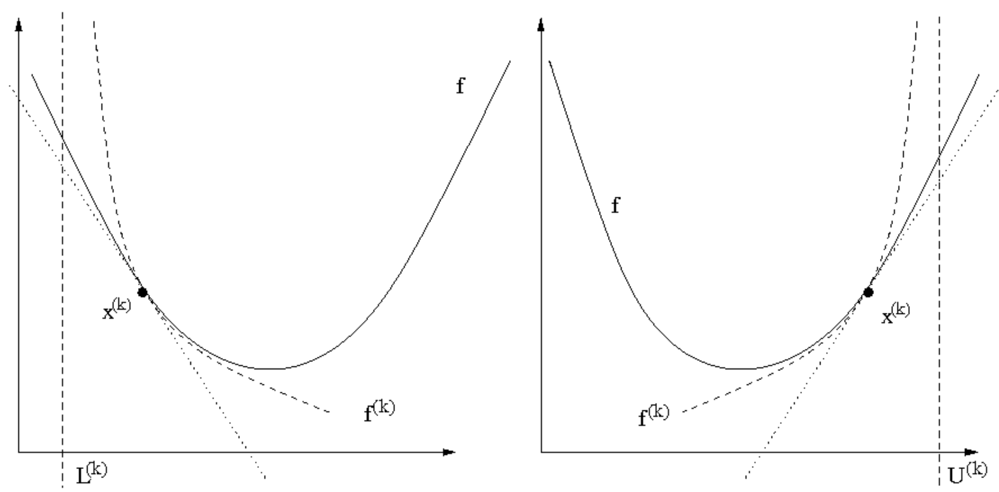

Figure 2: Method of Moving Asymptotes, Ertel [5].

Sequential convex programming methods are extensions of the method of moving asymptotes (MMA). They contain an additional merit function and a corresponding line-search algorithm. A possible merit function of (P) is the augmented Lagrangian function Φ_ρ for a given set of penalty parameters $\rho_j > 0$, $j = 1$,

..., m_c,

$$\Phi_\rho(x,y) = f(x) + \sum_{j=1}^{m_{\rm eq}} (y_j c_j(x) + \frac{\rho_j}{2} c_j^2(x)) + \sum_{j=m_{\rm eq}+1}^{m_c} \begin{cases} y_j c_j(x) + \frac{\rho_j}{2} c_j^2(x), & \text{if } -\frac{y_j}{\rho_j} \le c_j(x) \\ -\frac{y_j^2}{2\rho_j}, & \text{else,} \end{cases}$$

see also Schittkowski [16] for its usage as part of an SQP method. The merit function combines the objective function and constraints controlled by penalty parameters ρ_j, which must be carefully adapted by the algorithm, and possesses the following basic properties:

1. A point $(x^\star, y^\star)$ is a stationary point for Φ_ρ, if and only if $(x^\star, y^\star)$ is stationary for (P).
2. Under certain conditions there exists $\overline{\rho} \in \mathbb{R}^m$, $0 < \overline{\rho}$, such that $x^\star$ is a local minimizer for $\Phi_\rho(x, y^\star)$ for all $\rho \ge \overline{\rho}$.

The SCP algorithm can be summarized as follows:

Step 0: Choose starting value x^0 and let $k := 0$.

Step 1: Determine L_i^k and U_i^k and compute $f^k(x^k)$, $c_j^k(x^k)$, $j = 1, \ldots, m_c$.

Step 2: Solve $(P_{\rm sub}^k)$. Let (z^k, v^k) a KKT-point, where v^k is the multiplier vector.

Step 3: If $z^k = x^k$, then stop. $(x^k, y^k)^T$ is considered as a solution.

Step 4: Let $p^k := (z^k - x^k, v^k - y^k)$.

Step 5: Perform line search with respect to $\Phi(x^k, y^k)$ along direction p^k to get a steplength σ_k with a sufficient reduction of the merit function.

Step 6: Let $(x^{k+1}, y^{k+1}) = (x^k, y^k) + \sigma_k p^k$ and set $k := k + 1$.

Step 7: Compute $\nabla f(x^k)$, $\nabla c_j(x^k)$, $j = 1, \ldots, m_c$, and goto *Step 1.*

3 Free Material Optimization (FMO)

3.1 The General FMO Problem

Free material optimization (FMO) is an extension of topology optimization, where we try to find the optimal material distribution in form of elasticity tensors, see Bendsøe and Sigmund [4] and Zowe, Kočvara, and Bendsøe [26] for a more extensive description. FMO tries to find the *best* mechanical structure with respect to one or more given loads in the sense of minimal weight, maximal stiffness, or any other criterion. The material itself, as well as its distribution in the available space, can be varied.

A large number of different formulations of FMO problems is found in Kočvara et al. [9]. Depending on material properties, the number of load cases, the desired design criterion and the imposed constraints, there are nonlinear nonconvex semidefinite programs, nonlinear convex semidefinite programs, linear semidefinite programs, or nonconvex quadratic semidefinite programs. In this report, we only describe nonlinear, nonconvex FMO problems. In the other situations mentioned, alternative algorithms are available based on different mathematical methods.

As shown by Kočvara and Stingl [10], the general FMO problem can be formulated by a nonlinear, nonconvex semidefinite program (NSDP), where L load cases are taken into account. Design variables are matrices E_i that represent material properties, i.e., the elasticity tensors, which are symmetric and positive semidefinite for $i = 1, \ldots, m$, m number of elements of the discretized mechanical structure,

$$
\text{(NSDP)} \begin{cases} \min \sum_{i=1}^{m} \operatorname{Trace}(E_i) & \\ \text{s.t. } E_i \succeq 0 \, , & i = 1, \ldots, m \\ \quad \underline{\rho} \leq \operatorname{Trace}(E_i) \leq \overline{\rho} \, , & i = 1, \ldots, m \, , \\ \quad f_l^T K(E)^{-1} f_l^T \leq \gamma \, , & l = 1, \ldots, L \\ \quad E = (E_1, \ldots, E_m) \in \mathbb{R}^{n(d \times d)} & \end{cases}
$$

where

$$
K(E) = \sum_{i=1}^{m} K_i(E) = \sum_{i=1}^{m} \sum_{j=1}^{n_{ig}} B_{ij}^T E_i B_{ij}
$$

is the global stiffness matrix and where $E = (E_1, \ldots, E_m)$ denotes the set of all elasticity matrices. In the two dimensional space, each E_i is a 3×3 matrix, while in the three dimensional space we get 6×6 matrices, see Zowe, Kočvara, and Bendsøe [26], i.e.,

$$
E_i = \begin{bmatrix} E_{i1111} & E_{i1122} & \sqrt{2}E_{i1112} \\ E_{i1122} & E_{i2222} & \sqrt{2}E_{i2212} \\ \sqrt{2}E_{i1112} & \sqrt{2}E_{i2212} & 2E_{i1212} \end{bmatrix}
$$

or

$$
E_i = \begin{bmatrix} E_{i1111} & E_{i1122} & E_{i1133} & \sqrt{2}E_{i1112} & \sqrt{2}E_{i1113} & \sqrt{2}E_{i1123} \\ E_{i1122} & E_{i2222} & E_{i2233} & \sqrt{2}E_{i2212} & \sqrt{2}E_{i2213} & \sqrt{2}E_{i2223} \\ E_{i1133} & E_{i2233} & E_{i3333} & \sqrt{2}E_{i3312} & \sqrt{2}E_{i3313} & \sqrt{2}E_{i3323} \\ \sqrt{2}E_{i1112} & \sqrt{2}E_{i2212} & \sqrt{2}E_{i3312} & 2E_{i1212} & 2E_{i1213} & 2E_{i1223} \\ \sqrt{2}E_{i1113} & \sqrt{2}E_{i2213} & \sqrt{2}E_{i3313} & 2E_{i1213} & 2E_{i1313} & 2E_{i1323} \\ \sqrt{2}E_{i1123} & \sqrt{2}E_{i2223} & \sqrt{2}E_{i3323} & 2E_{i1223} & 2E_{i1323} & 2E_{i2323} \end{bmatrix}
$$

The indices follow the usual tensor notation.

To analyze the two systems in more detail, we assume that Hooke's law is valid, i.e., that stresses are linear functions of the strains. The constitutive equation in tensor notation is

$$
\underbrace{\sigma_{ij}}_{\text{stress}} = \underbrace{E_{ijkl}}_{\text{elasticity tensor}} \underbrace{\epsilon_{kl}(u)}_{\text{strain}} \, ,
$$

see Zowe, Kočvara, and Bendsøe [26], where $i, j, k, l \in \{1, 2\}$ in the 2-dimensional and $i, j, k, l \in \{1, 2, 3\}$ in the 3-dimensional case. Using the Einstein summation convention, we have

$$
\sigma_{ij} = E_{ijkl}\epsilon_{kl} = \sum_{k=1}^{3} \sum_{l=1}^{3} E_{ijkl}\epsilon_{kl}
$$

For σ_{11}, for example, we obtain after exchanging i by j, k by l and ij by kl

$$
\begin{aligned}
\sigma_{11} &= E_{11kl}\epsilon_{kl} = \sum_{k=1}^{3} \sum_{l=1}^{3} E_{11kl}\epsilon_{kl} \\
&= E_{1111}\epsilon_{11} + E_{1112}\epsilon_{12} + E_{1113}\epsilon_{13} + E_{1121}\epsilon_{21} + E_{1122}\epsilon_{22} + E_{1123}\epsilon_{23} + E_{1131}\epsilon_{31} \\
&\quad + E_{1132}\epsilon_{32} + E_{1133}\epsilon_{33} \\
&= E_{1111}\epsilon_{11} + 2E_{1112}\epsilon_{12} + 2E_{1113}\epsilon_{13} + E_{1122}\epsilon_{22} + 2E_{1123}\epsilon_{23} + E_{1133}\epsilon_{33}
\end{aligned}
$$

Analogously, we get similar expressions for $\sigma_{22}, \ldots, \sigma_{32}$. Furthermore, we know that $\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$ and $\sigma_{13} = \sigma_{31}$, and get the stress and strain tensors

$$\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \qquad \epsilon_{ij} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{21} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

In vector notation, we obtain

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 2E_{1112} & 2E_{1113} & 2E_{1123} \\ E_{1122} & E_{2222} & E_{2233} & 2E_{2212} & 2E_{2213} & 2E_{2223} \\ E_{1133} & E_{2233} & E_{3333} & 2E_{3312} & 2E_{3313} & 2E_{3323} \\ E_{1112} & E_{2212} & E_{3312} & 2E_{1212} & 2E_{1213} & 2E_{1223} \\ E_{1113} & E_{2213} & E_{3313} & 2E_{1213} & 2E_{1313} & 2E_{1323} \\ E_{1123} & E_{2223} & E_{3323} & 2E_{1223} & 2E_{1323} & 2E_{2323} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \epsilon_{12} \\ \epsilon_{13} \\ \epsilon_{23} \end{bmatrix}$$

To get a symmetric matrix, E_{ijkl}, $\sigma_{12}, \sigma_{13}, \sigma_{23}, \epsilon_{12}, \epsilon_{13}$ and ϵ_{23} are multiplied by $\sqrt{2}$, leading to

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sqrt{2}\sigma_{12} \\ \sqrt{2}\sigma_{13} \\ \sqrt{2}\sigma_{23} \end{bmatrix} = \begin{bmatrix} E_{i1111} & E_{i1122} & E_{i1133} & \sqrt{2}E_{i1112} & \sqrt{2}E_{i1113} & \sqrt{2}E_{i1123} \\ E_{i1122} & E_{i2222} & E_{i2233} & \sqrt{2}E_{i2212} & \sqrt{2}E_{i2213} & \sqrt{2}E_{i2223} \\ E_{i1133} & E_{i2233} & E_{i3333} & \sqrt{2}E_{i3312} & \sqrt{2}E_{i3313} & \sqrt{2}E_{i3323} \\ \sqrt{2}E_{i1112} & \sqrt{2}E_{i2212} & \sqrt{2}E_{i3312} & 2E_{i1212} & 2E_{i1213} & 2E_{i1223} \\ \sqrt{2}E_{i1113} & \sqrt{2}E_{i2213} & \sqrt{2}E_{i3313} & 2E_{i1213} & 2E_{i1313} & 2E_{i1323} \\ \sqrt{2}E_{i1123} & \sqrt{2}E_{i2223} & \sqrt{2}E_{i3323} & 2E_{i1223} & 2E_{i1323} & 2E_{i2323} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ \sqrt{2}\epsilon_{12} \\ \sqrt{2}\epsilon_{13} \\ \sqrt{2}\epsilon_{23} \end{bmatrix}$$

In the framework of PLATO-N, (NSDP) is to be solved by a new variant of the SCP algorithm handling semi-definite matrix variables and matrix constraints. The code SCPIP of Zillober [24] serves as our initial implementation and is to be gradually extended. The approaches considered for solving the primal program (NSDP), are illustrated in Figure 3 together with some others for solving the dual convex and linear semi-definite programs. These are:

1. SCPIP for non-convex problems with diagonally dominant matrix variables,
2. SCPIP for non-convex problems with positive semi-definite matrix variables,
3. Convex approximation of semi-definite matrix constraint,
4. CONERML (Technion) for convex problems with semi-definite matrix variables,
5. Linear approximation of semi-definite matrix constraint.

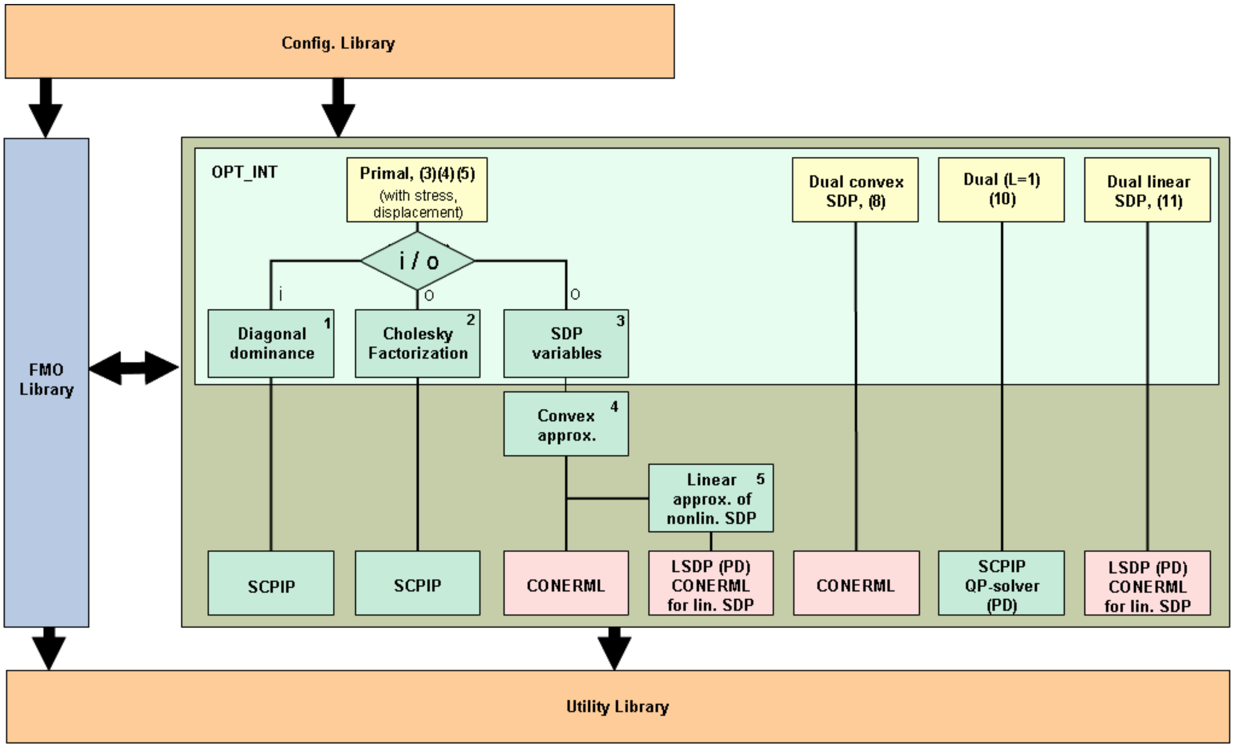

Figure 3: Approaches to solve FMO

In the subsequent sections, we consider only the 2-dimensional space to simplify the notation.

3.2 Cholesky Factorization of Positive Semi-Definite Matrix Variables

The idea of this approach is to generate an optimization problem without semi-definite matrix variables, which is directly solved by the code SCPIP. The constraints $E_i \succeq 0$, $i = 1, \ldots, m$, are omitted by optimizing directly the Cholesky factors of E_i, i.e., we assume that $E_i = D_i D_i^T$ and consider the lower triangular matrices $D_i \succeq 0$, $i = 1, \ldots, m$, as optimization variables. It is important to note that linear constraints in E_i become nonlinear in the new variables D_i.

Consider

$$D_i = \begin{bmatrix} d_{i1} & 0 & 0 \\ d_{i2} & d_{i3} & 0 \\ d_{i4} & d_{i5} & d_{i6} \end{bmatrix}$$

Then we get E_i by

$$E_i = \begin{bmatrix} d_{i1}^2 & d_{i1}d_{i2} & d_{i1}d_{i4} \\ d_{i2}d_{i1} & d_{i2}^2 + d_{i3}^2 & d_{i2}d_{i4} + d_{i3}d_{i5} \\ d_{i1}d_{i4} & d_{i2}d_{i4} + d_{i3}d_{i5} & d_{i4}^2 + d_{i5}^2 + d_{i6}^2 \end{bmatrix}$$

The FMO problem (NSDP) is now replaced by a nonlinear Cholesky decomposition approach (NCDA)

$$
\text{(NCDA)}\begin{cases}
\min \sum_{i=1}^{m}\sum_{j=1}^{6} d_{ij}^2 & \\
\text{s.t. } \underline{\rho} \le \sum_{j=1}^{6} d_{ij}^2 \le \overline{\rho}\ , & i=1,\dots,m\\
f_l^T K(D)^{-1} f_l^T \le \gamma\ , & l=1,\dots,L\\
d_{ij}\in\mathbb{R}\ , & i=1,\dots,m, j=1,\dots,6
\end{cases}
$$

where

$$
K(D)=\sum_{i=1}^{m} K_i(D)=\sum_{i=1}^{m}\sum_{j=1}^{n_{ig}} B_{ij}^T D_i D_i^T B_{ij}
$$

and $D=(D_1,\dots,D_m)$ with lower triangular matrices D_i. n_{ig} is the number of Gaussian integration points. See Kočvara et al. [9] for a more detailed definition of the block matrices B_{ij}. The general assumption that elasticity tensors have to be positive semi-definite, permits anisotropic materials.

3.3 Diagonal Dominant Matrix Variables

For isotropic materials, elasticity tensors E_i have a more restrictive structure, see Pedersen [15],

$$
E_i=\begin{bmatrix} E_{i1111} & E_{i1122} & 0\\ E_{i1122} & E_{i1111} & 0\\ 0 & 0 & E_{i1111}-E_{i1122}\end{bmatrix}
$$

These matrices must be positive semi-definite as before, but in addition diagonally dominant. Due to the special structure of E_i, both assumptions lead to the inequality constraints

$$
\begin{aligned}
E_{i1111} &\ge 0\\
E_{i1111} &\ge |E_{i1122}|
\end{aligned}
$$

To get twice differentiable constraints, the absolute values are replaced by two linear inequality constraints,

$$
\begin{aligned}
E_{i1111} &\ge 0\\
E_{i1111} &\ge -E_{i1122}\\
E_{i1111} &\ge E_{i1122}
\end{aligned}
$$

Thus, we get an alternative, simpler optimization problems called the nonlinear diagonally dominant approach,

$$
\text{(NDDA)}\begin{cases}
\min \sum_{i=1}^{m} 3E_{i1111}-E_{i1122} & \\
\text{s.t. } \underline{\rho}\le 3E_{i1111}-E_{i1122}\le\overline{\rho}\ , & i=1,\dots,m\\
f_l^T K(E)^{-1} f_l^T \le \gamma\ , & l=1,\dots,L\\
E_{i1111}\ge 0\ , & i=1,\dots,m\\
E_{i1111}+E_{i1122}\ge 0\ , & i=1,\dots,m\\
E_{i1111}-E_{i1122}\ge 0\ , & i=1,\dots,m\\
E_{i1111}, E_{i1122}\in\mathbb{R} & i=1,\dots,m
\end{cases}
$$

where

$$K(E) = \sum_{i=1}^{m} K_i(E) = \sum_{i=1}^{m}\sum_{j=1}^{n_{ig}} B_{ij}^T E_i^T B_{ij}$$

and $E = (E_1, \ldots, E_m)$ with above matrices E_i. n_{ig} is the number of Gaussian integration points. Again, see Kočvara et al. [9] for a more detailed definition of the block matrices B_{ij}. The resulting problem possesses a large number of sparse linear constraints, and is to be solved by SCPIP.

4 Displacement, Stress, and Stability Constraints

By additional constraints, bounds for displacements, stresses, vibrations and buckling are defined depending on the underlying FMO formulation.

Linear constraints on displacements can be imposed in selected areas in terms of

$$CK(E)^{-1} f^l \leq d \ ,$$

$l = 1, \ldots, L$. The matrix C serves to select nodes or combination of nodes for which displacements are to be restricted.

Stress constraints are highly important from the engineering point of view. However, exact formulation is still difficult. Stress constraints sometimes lead to optimization problems that do not satisfy the constraint qualification, see Kočvara and Stingl [10]. Local bounds can be declared in form of Kočvara et al. [9],

$$\sum_{k=1}^{n_{ig}} \| E B_{ik} K(E)^{-1} f^l \|_H^2 \leq s_\sigma \overline{\rho}^2 \ ,$$

$i = 1, \ldots, m$, $l = 1, \ldots, L$.

Vibrations lead to linear stability constraints of the form $K(E) + \hat{\lambda} M(E) \succeq 0$ with the mass matrix

$$M(E) = \sum_{i=1}^{m} \mathrm{Trace}(E_i) M_i$$

and $M_i = P_i \hat{M}_i P_i^T$. The matrices M_i are positive semidefinite and have the same nonzero structure as the element stiffness matrices K_i, see Kočvara et al. [9] for details.

The stability constraints for avoiding buckling require $K(E) + G(E_i) \succeq 0$ for all load cases and elements, $l = 1, \ldots, L$, $i = 1, \ldots, m$. $G(E_i)$ is the geometry stiffness matrix, see Kočvara and Stingl [13], defined by

$$G(E_i) := P^T \left(\sum_{i=1}^{m} \sum_{k=1}^{n_{ig}} Q_{ik}^T S_{ik}(E_i) Q_{ik} \right) P$$

with a permutation matrix P. Again, see Kočvara et al. [9] for more technical details and a more precise definition of the matrices Q_{ik} and $S_{ik}(E_i)$.

Whereas displacement and stress constraints can be added to the nonlinear programs (NSDP), (NCDA), or (NDDA) directly, vibration and buckling constraints lead to nonlinear nonconvex and semidefinite matrix restrictions. At the moment, some alternative methods to take them into account, are under consideration. A possible approach is to approximate these matrices in a suitable way to get convex matrix constraints. By successive solution of approximated convex nonlinear programs, a solution of the non-convex problem is obtained.

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